J Math Chem (2012) 50:2427–2438
DOI 10.1007/s10910-012-0040-8

ORIGINAL PAPER

Analytic solution of nonlinear singularly perturbed initial value problems through iteration

Aditya Kaushik · Ved Prakash Kaushik

Received: 15 February 2012 / Accepted: 30 May 2012 / Published online: 24 June 2012

Abstract This paper is concerned with singularly perturbed initial value problems for systems of ordinary differential equations. Here our emphasis will be on nonlinear phenomena and properties, particularly those with physical relevance. Since very few nonlinear systems can be solved explicitly, one must typically rely on a numerical scheme to accurately approximate the solution. However, numerical schemes do not always give accurate results, and we discuss the class of stiff differential equations, which present a more serious challenge to numerical analysts. In this paper, we derive in closed from, analytic solution of stiff nonlinear initial value problems, through iteration. The obtained sequence of iterates is based on the use of Lagrange multipliers. Moreover, the illustrative examples shows the efficiency of the method.

Keywords Variation iteration · Nonlinear initial value problem · Lagrange multiplier

1 Introduction

We consider the following one dimensional nonlinear singularly perturbed initial value problem:

$$\begin{aligned} &\epsilon y' + a(t)y(t) = f(t, y(t)) \quad \text{in } \Omega, \\ &y(t_0) = y_0. \end{aligned}$$

A. Kaushik
University Institute of Engineering and Technology, Panjab University, Chandigarh 160014, India
e-mail: akaushik@pu.ac.in

V. P. Kaushik (✉)
Department of Mathematics, Kurukshetra University, Kurukshetra 136119, Haryana, India
e-mail: vpkaushik@kuk.ac.in

Here $\Omega \equiv (0, 1)$ is the bounded domain with α the initial given data and $f(t)$ be the given nonlinear source function.

Mathematical models that involve initial value problems are quite important in all of science, engineering, and other fields where mathematical modeling is required. Very often the dimensionless parameter that measures the relative strength of the highest order derivative term is quite small. This implies that thin boundary and interior layers are present in the solution and singular perturbation problems arise. This kind of problem appears, for example, in the dynamics of chemical reactions [1–5], fluid or gas dynamics [6,7], heat transfer [8], theory of plates and shells [9], magnetohydrodynamic flow [10], or neuron variability [11–13]. An extensive selection of such type of problems of the physics or engineering may be found in [14]: pollutant dispersal in a river estuary, vorticity transport in the incompressible Navier-Stokes equations, atmospheric pollution, groundwater transport, turbulence transport, etc.

For small values of perturbation parameter, the use of standard higher order methods like the Galerkin finite elements or central differencing on uniform meshes leads to nonphysical oscillations in the computed solution. This is due to a loss in stability—unless the mesh diameter is extremely small, which is computationally expensive. Many authors studied such type of problems and proposed adaptive numeric or asymptotic techniques, such as non-conforming finite elements [15], monotone difference methods [16], local projection stabilization [17], streamline diffusion methods [18], fitted schemes [19,20], finite volume approximations [21] or weighted schemes [22].

The objective of this paper is to derive in closed form, the solutions of nonlinear singularly perturbed initial value problems, using variation iteration method [23]. The method was proposed originally by He [23] and is based on a Lagrange multiplier technique developed by Inokuti et al. [24] whereby the Lagrange multiplier is not a constant but a function. In their paper, Inokuti et al. [24] construct adjoint operators and sate that "λ (the Lagrange multiplier) may be regarded as a Greens function". In [25], it was shown that this claim is indeed correct and also show that Inokuti et al.s variational technique [24] and, therefore, Hes variational iteration method [23,26] can be derived by means of adjoint operators, Greens function, integration by parts and the method of weighted residuals. An elementary introduction to the variational iteration method and some new developments, as well as to new interpretations, can be found in [27–29]. There, the main concepts underlying the variational iteration method, such as the role of general Lagrange multipliers, the restricted variation and correction functionals are explained heuristically.

2 Variation iteration method

In this section, we present briefly the basic idea underlying the variational iteration method. Consider the following nonlinear equation:

$$\mathcal{L}y(t) \equiv L(y(t)) + N(y(t)) = f(t), \tag{2.1}$$

where L is a linear operator, N a nonlinear operator and $f(t)$ a known analytic inhomogeneous term. According to the variation iteration method, we can construct correction functional as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(s)\{Ly_n(s) + N\tilde{y}_n(s) - f(s)\}ds, \quad n \geq 0 \tag{2.2}$$

where λ is a general Lagrange's multiplier, which can be identified optimally via variational theory and using integration by parts. Here, y_n $(n \geq 0)$ be the approximate solution at nth iteration and $\tilde{y}_n$ denotes the restricted variation which means $\delta\tilde{y}_n = 0$. After determining the Lagrange multiplier λ (in subsequent discussion, it is to be noted that the Lagrange multiplier can be a constant or a function) and selecting an appropriate initial function y_0, (any function that satisfies the boundary condition) the successive approximations y_n of the solution y can be readily obtained using (2.2). Consequently, the exact solution of the problem may be obtained using $y = \lim_{n\to\infty} y_n$. Let us elucidate this idea further with the help of a linear example.

Example 2.1 Single-species models are of relevance to laboratory studies in particular but, in the general world, can reflect a telescoping of effects which influence the population dynamics. Let $y(t)$ be the population of the species at time t, then a linear model of population growth together with the prescribed boundary conditions reads

$$\epsilon\frac{dy(t)}{dt} = (2t-1)y \quad t \in (0,1); \quad y(0) = A. \tag{2.3}$$

where ϵ is very small. Exact solution of the problem reads,

$$y(t) = A\exp\frac{t(t-1)}{\epsilon} = 0.5\exp\frac{t(t-1)}{\epsilon} \quad \text{for A} = 0.5. \tag{2.4}$$

The correction function corresponding to (2.3) can be written as

$$\begin{aligned} y_{n+1}(t) &= y_n(t) + \int_0^t \lambda(s)\left[\frac{dy_n(s)}{ds} - \frac{2s-1}{\epsilon}y_n(s)\right]ds \\ &= y_n(t) + \lambda(s)y_n(s)|_{s=t} - \int_0^t \left(\frac{d\lambda(s)}{ds} + \lambda(s)\frac{2s-1}{\epsilon}\right)y_n(s)ds. \end{aligned} \tag{2.5}$$

In order to determine Lagrange's multiplier λ optimally, we make use of variational theory. Taking variation with respect to the independent variable y_n (notice that $\delta y_n(0) = 0$) and making the correctional functional (2.5) stationary i.e. $\delta y_{n+1} = 0$:

$$\delta y_{n+1}(t) = \delta y_n(t) + \lambda(s)\delta y_n(s)|_{s=t} - \int_0^t \left(\frac{d\lambda(s)}{ds} + \lambda(s)\frac{2s-1}{\epsilon}\right)\delta y_n(s)ds. \tag{2.6}$$

Hence, the Euler-Lagrange equation reads

$$\frac{d\lambda(s)}{ds} + \lambda(s)\frac{2s-1}{\epsilon} = 0 \tag{2.7}$$

and the stationary condition so obtained is

$$1 + \lambda(s)|_{s=t} = 0. \tag{2.8}$$

The Lagrange multiplier can readily be defined as a solution of (2.7–2.8) given by

$$\lambda(s) = -\exp\frac{(t-s)(t+s-1)}{\epsilon}.$$

As a result, iteration formula becomes

$$y_{n+1}(t) = y_n(t) - \int_0^t \exp\frac{(t-s)(t+s-1)}{\epsilon}\left[\frac{dy_n(s)}{ds} - \frac{2s-1}{\epsilon}y_n(s)\right]ds. \tag{2.9}$$

Starting with $y_0(t) = 0.5$ and using the iteration formula (2.9), we find

$$\begin{aligned} y_1(t) &= 0.5\left(1 - \int_0^t \exp\frac{(t-s)(t+s-1)}{\epsilon}\left[\frac{1-2s}{\epsilon}\right]ds\right) \\ &= 0.5\left(1 - \left[\exp\frac{t^2}{\epsilon} - \frac{s^2}{\epsilon} - \frac{t}{\epsilon} + \frac{s}{\epsilon}\right]_0^t\right) \\ &= 0.5\exp\frac{t(t-1)}{\epsilon}. \end{aligned} \tag{2.10}$$

Clearly, $y_1(t)$ coincides with the exact solution (2.4).

3 Nonlinear initial value problems

3.1 Statement of the problem and auxiliary results

Consider the following nonlinear initial value problem:

$$\left.\begin{array}{l} \epsilon y' + a(t)y(t) = f(t, y(t)) \quad \text{in } \Omega, \\ y(0) = y_0, \end{array}\right\} \tag{3.1}$$

where ϵ is a small parameter and y_0 be the given initial data. The functions $a(t)$ is assumed to be sufficiently smooth and the nonlinear source term $f(t, y(t))$ is assumed to be sufficiently differentiable function so that

$$\alpha \leq \frac{\partial f}{\partial y} \leq \alpha^* < \infty.$$

By virtue of Mean-Value Theorem, Eq. (3.1) can be written as

$$\epsilon y' + (a(t) + b(t))u = f(t, 0), \tag{3.2}$$

where $b(t) = -\frac{\partial f}{\partial y}(t, \tilde{y})$, $\tilde{y} = \gamma y$, $0 < \gamma < 1$. Now, multiply both sides of Eq. (3.2) by $\frac{1}{\epsilon} \exp\left(\frac{1}{\epsilon}\int_0^t (a(s) + b(s))ds\right)$, then we have

$$\left(y \exp\left(\frac{1}{\epsilon}\int_0^t (a(s) + b(s))ds\right)\right)' = \frac{f(t, 0)}{\epsilon} \exp\left(\frac{1}{\epsilon}\int_0^t (a(s) + b(s))ds\right).$$

Integration within the limits 0 to t gives

$$\begin{aligned} y(t) = {} & y_0 \exp\left(-\frac{1}{\epsilon}\int_0^t (a(s) + b(s))ds\right) \\ & + \frac{1}{\epsilon}\int_0^t f(\tau, 0) \exp\left(-\frac{1}{\epsilon}\int_\tau^t (a(s) + b(s))ds\right) d\tau. \end{aligned}$$

This implies

$$\begin{aligned} |y(t)| \leq {} & |y_0| \left|\exp\left(-\frac{1}{\epsilon}\int_0^t (a(s) + b(s))ds\right)\right| \\ & + \frac{1}{\epsilon}|f(\tau, 0)| \left|\int_0^t \exp\left(-\frac{1}{\epsilon}\int_\tau^t (a(s) + b(s))ds\right) d\tau\right| \\ \leq {} & |y_0| \exp\left(-\frac{\beta t}{\epsilon}\right) + \beta^{-1}\left(1 - \exp\left(-\frac{\beta t}{\epsilon}\right)\right) |f(\tau, 0)|. \end{aligned}$$

Therefore, $\|y\|_\infty \leq C_1$. Now, $|y'(0)| \leq \frac{1}{\epsilon}(|f(0, \alpha)| + |a(t)||u(t)|)$. Therefore,

$$|y'(0)| \leq \frac{C}{\epsilon}. \tag{3.3}$$

Differentiating of Eq. (3.1) yields

$$\epsilon y'' + (a(t) + b(t))u' = -\phi(x) - a'(t)u(t), \tag{3.4}$$

where $b(t) = -\frac{\partial f}{\partial y}(t, y)$ and $\phi(t) = -\frac{\partial f}{\partial t}(t, y)$. Again multiplying both sides of the Eq. (3.4) by $\frac{1}{\epsilon}\exp\left(\frac{1}{\epsilon}\int_0^t (a(s)+b(s))ds\right)$, we get

$$\left(y'(t)\exp\left(\frac{1}{\epsilon}\int_0^t (a(s)+b(s))ds\right)\right)' = -\frac{\phi(t)+a(t)u(t)}{\epsilon}\exp\left(\frac{1}{\epsilon}\int_0^t (a(s)+b(s))ds\right).$$

Now integrating both sides within the limits 0 to t, we get

$$y'(t) = y'(0)\exp\left(-\frac{1}{\epsilon}\int_0^t (a(s)+b(s))ds\right) - \frac{1}{\epsilon}\int_0^t (\phi(\tau)+a'(\tau)u(\tau))\exp\left(-\frac{1}{\epsilon}\int_\tau^t (a(s)+b(s))ds\right)d\tau.$$

This implies

$$|y'(t)| \leq \frac{C}{\epsilon}\exp\left(-\frac{\beta t}{\epsilon}\right) + (|\phi(\tau)| + |a'(\tau)||u(\tau)|)\beta^{-1}\left(1-\exp\left(-\frac{\beta t}{\epsilon}\right)\right) \leq C\left(\frac{1}{\epsilon}\exp\left(-\frac{\beta t}{\epsilon}\right) + \beta^{-1}\left(1-\exp\left(-\frac{\beta t}{\epsilon}\right)\right)\right).$$

Hence, the solution of the problem 3.1 and its derivative are bounded.

3.2 Variation iteration method

The correctional function corresponding to (3.1) can be written as

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(s)\left[\frac{dy_n(s)}{ds} + \frac{a(s)}{\epsilon}y_n(s) - \frac{1}{\epsilon}\tilde{f}(s, y_n(s))\right]ds, \quad (3.5)$$

where $\lambda(s) \equiv \lambda(s, t)$ is the general Lagrange multiplier which is to be determined and $\tilde{f}(s, y_n(s))$ denotes restricted variation i.e. $\delta\tilde{f}(s, y_n(s)) = 0$.

In order to determine Lagrange's multiplier λ optimally, we make use of variational theory. Taking variation with respect to the independent variable y_n (notice that $\delta y_n(0) = 0$) and making the correctional functional (3.5) stationary i.e.$\delta y_{n+1} = 0$:

$$\delta y_{n+1}(t) = \delta y_n(t) + \lambda(s)\delta y_n(s)|_{s=t} - \int_0^t \left(\frac{d\lambda(s)}{ds} - \lambda(s)\frac{a(s)}{\epsilon}\right)\delta y_n(s)ds. \quad (3.6)$$

Therefore, Euler-Lagrange equation becomes

$$\frac{d\lambda(s)}{ds} - \lambda(s)\frac{a(s)}{\epsilon} = 0 \quad (3.7)$$

and the stationay condition so obtained reads

$$1 + \lambda(s)|_{s=t} = 0. \quad (3.8)$$

The Lagrange multiplier can readily be identified as a solution of (3.7–3.8) given by

$$\lambda(s) = -\exp\left(-\int \frac{a(t)}{\epsilon}dt + \int \frac{a(s)}{\epsilon}ds\right).$$

As a result, iteration formula becomes

$$y_{n+1}(t) = y_n(t) + \int_0^t -\exp\left(-\int \frac{a(t)}{\epsilon}dt + \int \frac{a(s)}{\epsilon}ds\right)\left[\frac{dy_n(s)}{ds} + \frac{a(s)}{\epsilon}y_n(s) - \frac{1}{\epsilon}f(s, y_n(s))\right]ds. \quad (3.9)$$

4 Numerical illustrations

In this section, we will apply the variational iteration method to

Example 4.1 Consider the initial value problem (3.1) with $a(t) = 0$ and $f = y^2$. Exact solution of the problem with $y(t_0) = y_0$ reads

$$y(t) = -\frac{\epsilon}{t - \epsilon\left(\frac{1}{y_0} + \frac{t_0}{\epsilon}\right)} = \frac{\epsilon y_0}{\epsilon - y_0(t - t_0)}. \quad (4.1)$$

Let us solve it using variation iteration formula and see if the proposed iterative method is qualitatively consistent with the way exact solution behaves. To this effort, we have from (3.9):

$$y_{n+1}(t) = y_n(t) - \int_0^t \left[\frac{dy_n(s)}{ds} - \frac{1}{\epsilon}u_n^2\right]ds.$$

Starting with $y(0) = y_0$,we find

$$y_1(t) = y_0 - \int_0^t y_0' - \frac{1}{\epsilon} y_0^2 \, ds = y_0 \left(1 + \frac{y_0}{\epsilon} t\right).$$

$$y_2(t) = y_0 + \frac{y_0^2}{\epsilon} t - \int_0^t \frac{y_0^2}{\epsilon} - \frac{1}{\epsilon}\left(y_0 + \frac{y_0^2}{\epsilon} s\right)^2 ds$$

$$= y_0 \left(1 + \frac{y_0}{\epsilon} t + \frac{y_0^2}{\epsilon^2} t^2\right) + \mathcal{O}\left(\frac{t^3}{\epsilon^3}\right).$$

$$y_3(t) = y_0 \left(1 + \frac{y_0}{\epsilon} t + \frac{y_0^2}{\epsilon^2} t^2\right) - \int_0^t \frac{y_0^2}{\epsilon^2} + \frac{2y_0^3}{\epsilon^2} s - \frac{y_0^2}{\epsilon}\left(1 + \frac{y_0}{\epsilon} s + \frac{y_0^2}{\epsilon^2} s^2\right)^2 ds,$$

$$= y_0 \left(1 + \frac{y_0}{\epsilon} t + \frac{y_0^2}{\epsilon^2} t^2 + \frac{y_0^3}{\epsilon^3} t^3\right) + \mathcal{O}\left(\frac{t^4}{\epsilon^4}\right) \overset{n\to\infty}{\longrightarrow} \frac{\epsilon y_0}{\epsilon - y_0 t}.$$

Clearly, as t approaches the critical value $t^* = t_0 + \epsilon/y_0$ from below, the solution blows up, meaning $u(t) \to \infty$ as $t \to t^*$. The blow up time t^* depends upon the initial data: the larger $y_0 > 0$ is, the sooner the solution goes off to infinity (Fig. 1). If the initial data is negative, $y_0 < 0$, the solution is well-defined for all $t > t_0$ (Fig. 2), but has a singularity in the past, at $t^* = t_0 + \epsilon/y_0$. The only solution that exists for all positive and negative time is the constant solution $y(t) = 0$, corresponding to the initial condition $y_0 = 0$. Clearly, the limiting solution coincides with the exact solution.

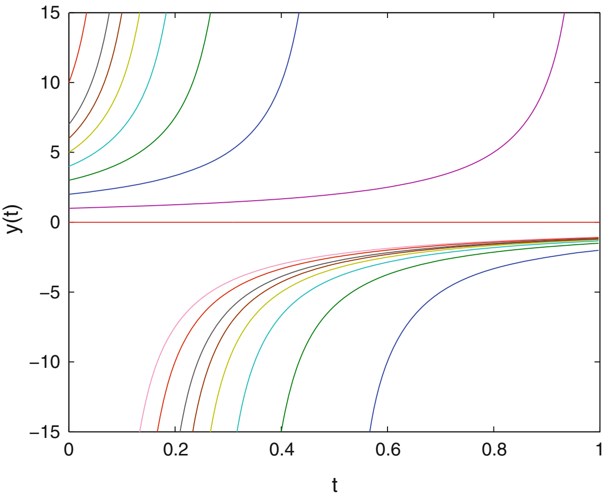

Fig. 1 Some typical solutions of Example 4.1 for different values of $y_0 \geq 0$

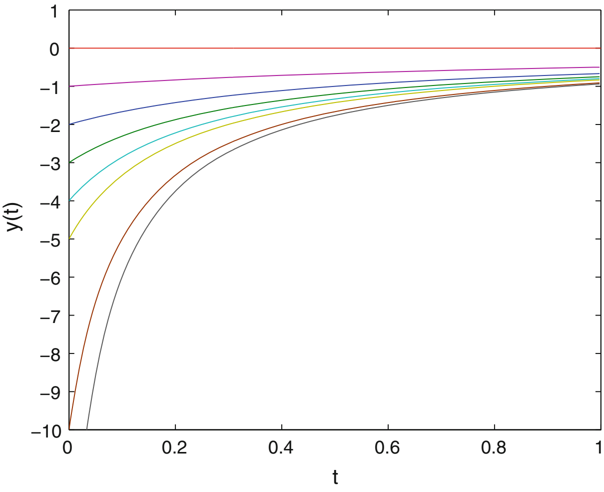

Fig. 2 Some typical solutions of Example 4.1 for different values of $y_0 \leq 0$

Example 4.2 Consider the initial value problem (3.1) with $a(t) = -\lambda$ and $f = -\lambda y^2$. Exact solution of the problem with $y(0) = y_0$ reads

$$y(t) = -\frac{y_0 \exp(\lambda t/\epsilon)}{1 - y_0 + y_0 \exp(\lambda t/\epsilon)} = -\frac{y_0 \exp(t/\epsilon)}{1 - y_0 + y_0 \exp(t/\epsilon)} \quad (\text{for } \lambda = 1). \quad (4.2)$$

When using the so-called logistic equation to model population dynamics, the initial data is assumed to be positive, $y_0 > 0$. As time $t \to \infty$ the solution 4.2 tends to the equilibrium value $y(t) = 1$. For small initial values $y_0 << 1$ the solution initially grows at an exponential rate λ, corresponding to a population with unlimited resources. However, as the population increases, the gradual lack of resources tends to slow down the growth rate, and eventually the population saturates at the equilibrium value. On the other hand, if $y_0 > 1$, the population is too large to be sustained by the available resources, and so dies off until it reaches the same saturation value. If $y_0 = 0$, then the solution remains at equilibrium $y(t) \equiv 0$. Finally, when $y_0 < 0$, the solution only exists for a finite amount of time, with

$$y(t) \to -\infty \quad \text{as} \quad t \to t^* = \frac{1}{\lambda} \log\left(1 - \frac{1}{y_0}\right).$$

Of course, this final case does appear in the physical world, since we cannot have a negative population!, so this may be discared. Let us solve it using variation iteration formula and see if the proposed iterative method is qualitatively consistent with the way exact solution behaves. To this effort, we have from (3.9):

$$y_{n+1}(t) = y_n(t) - \int_0^t \exp\left(\frac{t-s}{\epsilon}\right)\left[\frac{dy_n(s)}{ds} - \frac{1}{\epsilon}(y_n - y_n^2)\right] ds.$$

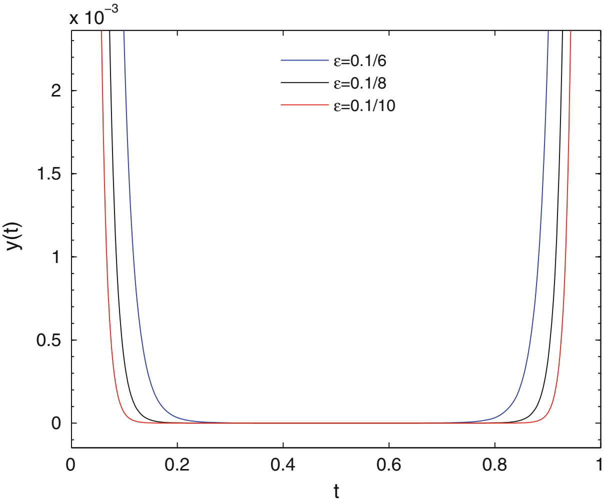

Fig. 3 Some typical solutions of Example 2.1 for different values of ϵ

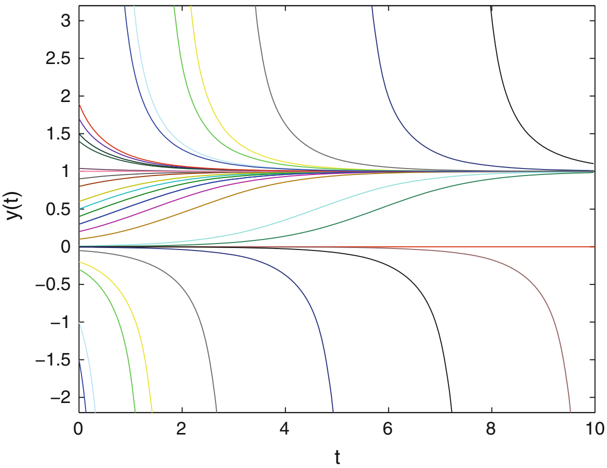

Fig. 4 Some typical solutions of Example 4.2 for different values of y_0

Starting with $y(0) = y_0$, we find

$$y_1(t) = y_0 - \int_0^t \exp\left(\frac{t-s}{\epsilon}\right)\left[\frac{dy_0(s)}{ds} - \frac{1}{\epsilon}(y_0 - y_0^2)\right] ds, = y_0 + \left(\frac{1}{\epsilon}y_0 - \frac{1}{\epsilon}y_0^2\right) t$$

$$y_2(t) = y_1 + \left(\frac{1}{2\epsilon^2}y_0 - \frac{1}{2\epsilon^2}y_0^2 + \frac{1}{\epsilon^2}y_0^2(-1+y_0)\right) t^2 + \mathcal{O}\left(\frac{t^3}{\epsilon^3}\right).$$

$$y_3(t) = y_2 + \left(\frac{-1}{6\epsilon^3}y_0^2 + \frac{1}{6\epsilon^3}y_0 + \frac{1}{2\epsilon^3}y_0^2(-1+y_0) - \frac{1}{2\epsilon^3}y_0^2(1-3y_0+2y_0^2)\right) t^3$$

$$+\mathcal{O}\left(\frac{t^4}{\epsilon^4}\right) \overset{n\to\infty}{\longrightarrow} -\frac{y_0 \exp(t/\epsilon)}{1-y_0+y_0\exp(t/\epsilon)}.$$

Clearly, the limiting solution coincides with the exact solution (Figs. 3, 4).

5 Conclusion

In this paper, we have demonstrated the applicability of variation iteration method for solving nonlinear singularly perturbed autonomous initial value problems. It is observed that the method is easy to implement. Moreover, it provides analytical approximations to both linear and nonlinear problems without any linearization and discretization. In case of linear problem (Example 2.1) the exact solution of the problem is obtained using one iteration only. It can be concluded that the variation iteration method is promising and readily implemented.

References

1. W. Benzinger, A. Becker, K.J. Httinger, Chemistry and kinetics of chemical vapour deposition of pyrocarbon: I. Fundamentals of kinetics and chemical reaction engineering. Carbon **34**, 957–966 (1996)
2. M. Danish, R.K. Sharma, S. Ali, Gas absorption with first order chemical reaction in a laminar falling film over a reacting solid wall. Appl. Math. Model. **32**, 901–929 (2008)
3. Y. Liu, L. Shen, A general rate law equation for biosorption. Biochem. Eng. J. **38**, 390–394 (2008)
4. Y. Liu, New insights into pseudo-second-order kinetic equation for adsorption. Colloids Surf. A **320**, 275–278 (2008)
5. C.V. Rao, D.M. Wolf, A.P. Arkin, Control, exploitation and tolerance of intracellular noise. Nature **420**, 231–237 (2002)
6. P.-C. Lu, *Introduction to the Mechanics of Viscous Fluids* (Holt, Rinehart and Winston, New York, 1973)
7. M. Van Dyke, *Perturbation Methods in Fluid Dynamics* (Academic Press, New York, 1964)
8. A. Bejan, *Convection Heat Transfer* (Wiley, New York, 1984)
9. J.K. Knowles, R.E. Messick, On a class of singular perturbation problems. J. Math. Anal. Appl. **9**, 42–58 (1964)
10. R.R. Gold, Magnetohydrodynamic pipe flow. Part I. J. Fluid Mech. **13**, 505–512 (1962)
11. A. Kaushik, Singular perturbation analysis of bistable differential equation arising in the nerve pulse propagation. Nonlinear Anal. **09**(5), 2106–2127 (2008)
12. A. Kaushik, Nonstandard perturbation approximation and traveling wave solutions of non-linear reaction diffusion equations. Numer. Methods Partial Differ. Equ. **24**(1), 217–238 (2008)
13. A. Kaushik, M.D. Sharma, Numerical Analysis of a mathematical model for propagation of an electrical pulse in a neuron. Numer. Methods Partial Differ. Equ. **27**, 1–18 (2008)
14. K.W. Mortan, *Numerical Solution of Convection Diffusion Problems* (Chapman & Hall, London, 1996)
15. O. Havle, V. Dolej$\check{s}\acute{i}$, M. Feistauer, Discontinuous Galerkin method for nonlinear convection diffusion problems with mixed Dirichlet-Neumann boundary conditions. Appl. Math. **55**(5), 353–372 (2010)
16. E. ORiordan, J. Quinn. Parameter uniform numerical methods for some linear and nonlinear singularly perturbed convection diffusion boundary turning point problems. BIT Numer. Math. (2010). doi:10.1007/s10543-010-0290-4
17. S. Franz, G. Matthies, Local projection stabilisation on S-type meshes for convection diffusion problems with characteristic layers. Computing **87**, 135–167 (2010)
18. S. Franz. SDFEM with non-standard higher order finite elements for a convection diffusion problem with characteristic boundary layers. BIT Numer. Math. (2011). doi:10.1007/s10543-010-0307-z
19. A. Kaushik, K.K. Sharma, M. Sharma, A parameter uniform difference scheme for parabolic partial differential equation with a retarded argument. Appl. Math. Model. **34**, 4232–4242 (2010)

20. A. Kaushik, V. Kumar, M. Sharma, Analysis of factorization method for elliptic differential equation. Comput. Math. Model. **22**(1), 98–110 (2011)
21. M. Ohlberger, A posteriori error estimate for finite volume approximations to singularly perturbed nonlinear convection diffusion equations. Numer. Math. **87**, 737–761 (2001)
22. A. Kaushik, M. Sharma. Convergence analysis of weighted difference approximations on piecewise uniform grids to a singularly perturbed functional differential equations. J. Optim. Theory Appl. doi:10.1007/s10957-011-9965-5
23. J.-H. He, Variational iteration method-A kind of non-linear analytical technique: some examples. Int. J. Non-Linear Mech. **34**(4), 699–708 (1999)
24. M. Inokuti, H. Sekine, T. Mura, in *General use of the Lagrange multiplier in nonlinear mathematical physics*, ed. by S. Nemat-Nasser Variational Methods in the Mechanics of Solids (Pergamon Press, New York, 1978), pp. 156–162
25. J.L. Ramos, On the variational iteration method and other iterative techniques for nonlinear differential equations. Appl. Math. Comput. **199**, 39–69 (2008)
26. J.-H. He, Variational iteration method—some recent results and new interpretations. J. Comput. Appl. Math. **207**(1), 3–17 (2007)
27. J.-H. He, X.-H. Wu, Variational iteration method: new development and applications. Comput. Math. Appl. **54**(7–8), 881–894 (2007)
28. J.-H. He, G. Wu, F. Austin, The variational iteration method which should be followed. Nonlinear Sci. Lett. A **1**(1), 1–30 (2010)
29. N. Herisanu, Vasile Marinca, A modified variational iteration method for strongly nonlinear problems. Nonlinear Sci. Lett. A **1**(2), 183–192 (2010)